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COUNTABLE MIXED ABELIAN GROUPS WITH VERY NICE FULL-RANK SUBGROUPS

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ABSTRACT

We exhibit a maximal set of $2^{\aleph_0}$ "almost rigid" countable mixed abelian groups G with the same prescribed torsion subgroups tG, the same quotient G/tG and a fixed countable and cotorsion-free ring A such that $\operatorname{End} G/\operatorname{Hom}(G, tG) \cong A$. Despite the fact that these candidates will not allow any structure theorem, they are close relatives of the well-behaving family of Warfield groups. The results are developed in a module category for suitable ground rings.

§1. Introduction

In this paper we will consider proper mixed R-modules of at most countable rank, over a fixed integral domain R. If tG is the torsion submodule of an R-module G, then *proper* denotes that tG is not a direct summand of G; and in particular $0 \neq tG \neq G$. The rank of a mixed module G will always be its torsion-free rank, which is the cardinal $\operatorname{rk} G = \dim_Q (G \otimes Q)$, where Q is the quotient field of R.

Restricting for a moment to abelian groups ($R = \mathbf{Z}$), we will provide and discuss some wild examples of countable, proper mixed abelian groups from an apparently well-behaving family close to Warfield groups: Starting from the well-known Ulm theorem, Kaplansky and Mackey [9] derived in 1951 a classification theorem for countably generated modules of rank one over a complete discrete valuation ring. Later, Megibben [12] showed that the ground ring of this class of modules need not be complete. This result was extended to the Hill–Walker Theorem, which can be stated as follows:

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Let H and H' be nice subgroups of the (abelian) groups G and G' respectively, such that G/H and G'/H' are totally projective with the same relative Ulm invariants. Then any height preserving isomorphism from H to H' extends to an isomorphism from G to G'.

Compare [13,14] or [15,16]. (Unexplained notions may be found in [7], Vol. II, Chap. XIV, and in the subsequent sections.) For the moment it suffices to know that totally projective groups are classified by invariants and that direct sums of cyclic torsion groups are very special totally projective groups. Recall that H is p-nice in G if $p^\sigma(G/H) = (p^\sigma G + H)/H$ for all ordinals σ. The rank one groups which are classified by the Hill–Walker Theorem are precisely the nice extensions of $\mathbf{Z}$ by the totally projective groups. This immediately leads to Warfield groups which are summands of totally projective groups: It is easy to see that G is a Warfield group iff it contains a decomposition basis X such that $\langle X \rangle$ is nice in G and $G/\langle X \rangle$ is totally projective; cf. [8], p. 118, Theorem 29. The set X is a *decomposition basis* iff $G/\langle X \rangle$ is torsion and $\langle X \rangle$ is a free *valuated* subgroup of G with basis X; hence $\langle X \rangle = \mathbf{Z}$ in the Hill–Walker Theorem. Warfield groups are classified by Ulm–Kaplansky invariants and Warfield invariants; compare [8], pp. 126–128, [13, 14] and [15, 16]. Hence Warfield groups are an obviously well-behaving family!

We will pick a family $\mathscr{E}$ of mixed groups very close to the class of Warfield groups: Let $E \in \mathscr{E}$ if E is a countable abelian group containing a nice torsion-free subgroup N (of finite rank) with E/N a direct sum of cyclic p-groups. Comparing with Warfield groups, we pose even stronger conditions on the quotients and replace the decomposition basis by "torsion-free (of finite rank)". We will show that there is no hope to classify $\mathscr{E}$-groups (even if $\mathscr{E}$ is restricted to finite rank N's). This will follow from a result on endomorphism rings, which answers a question asked recently by David Arnold.

The classification theorem mentioned above supports a result of May and Toubassi [11] which says that the endomorphism ring of a (mixed) rank one group determines the group. On the other hand there are many groups of uncountable rank with the same endomorphism ring; for rank $2^{\aleph_0}$ see [11] and [2] and for larger ranks compare [4] and [2]; the results in [2] and [4] are based on constructions in [6]. Hence it is interesting to investigate countable mixed groups and their endomorphism rings. We will proceed as follows.

Let A be a countable ring with 1 such that its additive group A^+ is torsion-free and reduced. By a well-known theorem of Corner [1] we find a countable and reduced, torsion-free abelian group N with endomorphism ring $\operatorname{End} N = A$. The rank of N may be related to the rank of A and in particular we may choose $\mathrm{rk}\, N$ to be finite iff the rank of the ring A is finite, cf. [1] or [7], Vol. II, p. 234. Then we can find a nice extension G of N such that $\mathrm{End}\, G = A \oplus \mathrm{Hom}(G, tG)$ and G/N is a countable direct sum of cyclic p-groups, i.e. $G \in \mathscr{C}$. The torsion subgroup tG will be a direct sum T of cyclic torsion subgroups with a divisible quotient G/tG of (if needed) fixed rank, The structure of the endomorphism ring implies the classical violations of the Krull–Schmidt Theorem, e.g. analogs to Kaplansky's test problems. The counter examples are similar to the one discussed in [5] and [6].

The given construction has a natural extension: We can find $2^{\aleph_0}$ such groups G with the same torsion subgroup T, the same quotient G/T and $\mathrm{End}\, G = A \oplus \mathrm{Hom}(G, T)$. If G and G' are two distinct extensions of this set, then $\mathrm{Hom}(G, G') = \mathrm{Hom}(G, T)$. In addition to this we also may fix the nice subgroup N, and variation of the height sequences responsible for the embedding $N \subseteq G$ leads to a similar family of mixed groups.

These results were stimulated by a question of David Arnold, who asked us for a realization theorem of rings on mixed, countable abelian groups. We were surprised to find the underlying groups in the very accessible class $\mathscr{C}$. This also fits nicely into the frame of a WALK-category. E. A. Walker defines the now-called *Walker* homomorphisms between two groups G and H to be

$$\mathrm{Walk}(G, H) = \mathrm{Hom}(G, H)/\mathrm{Hom}_t(G, H)$$

where $\mathrm{Hom}_t(G, H) = \mathrm{Hom}(G, tH)$. It turns out that G and H are *isomorphic in WALK* iff there exist torsion groups T and S with $G \oplus T \cong H \oplus S$, cf. [8], p. 87, Theorem 3. WALK is an additive category with infinite sums and kernels, which is important for the investigation of mixed groups; see the excellent articles [15, 16] and [13, 14]. We derive from our theorem above that all countable, torsion-free and reduced rings are realized as endomorphism rings of $\mathscr{C}$-groups in WALK; in fact we find a WALK-rigid system of maximal size $2^{\aleph_0}$ which realizes the given ring. This also contributes to problems #8, 9, pp. 30, 31 in [15].

§2. Preliminaries and statement of the main theorem

(2.1) We will assume throughout that R is a non-zero integral domain with 1 and a given *countable*, multiplicatively closed subset S which satisfies the Hausdorff condition $\bigcap_{s \in S} sR = 0$; as usual we require $1 \in S$. The S-topology on an R-module G has the countable set of submodules sG ($s \in S$) as a basis of neighbourhoods of 0. Then G is Hausdorff iff $\bigcap_{s \in S} sG = 0$ or, as we say, iff G is S-reduced. The S-torsion submodule of G is

$$tG = t_S G = \{g \in G : \exists s \in S \text{ with } sg = 0\}.$$

Then $\hat{G}$ is S-torsion iff $tG = G$ and S-torsion-free iff $tG = 0$. If G is S-reduced, we can build the S-completion G, that is, the completion of G in the S-topology. Then $\hat{G}$ is S-reduced and contains G as a S-pure submodule, i.e. $G \cap s\hat{G} = sG$ ($s \in S$); moreover G is S-dense in $\hat{G}$, i.e. $\hat{G}/G$ is S-divisible in the obvious sense. The module G is S-bounded if we find some $s \in S$ such that $sG = 0$.

(2.2) We will also assume throughout that A is a S-reduced and S-torsion-free R-module. Moreover, let N be a S-pure submodule of $\hat{A}$.

This condition arises naturally from a result of Corner [3]: If R in (2.1) is also countable and A is a countable, S-reduced and S-torsion-free R-algebra, then there exist countable, S-reduced and S-torsion-free R-modules G_i ($i \leqq 2^{\aleph_0}$) such that $A \subseteq G_i \subseteq \hat{A}$, $\operatorname{End} G_i = A$ and $\operatorname{Hom}(G_i, G_j) = 0$ ($i \neq j$).

If A has rank n, we may assume that the modules G_i have rank $\leqq 2n$, as follows from [10]. This result was published for abelian groups ($R = \mathbf{Z}$) in [1].

(2.3) *Very nice submodules.* A subgroup N of a p-local mixed abelian group G is said to be nice if $p^\sigma(G/N) = p^\sigma G + N$ for all ordinals σ. In particular N is nice in G, if G/N is a direct sum of bounded groups. The subgroup N is of full rank if N is torsion-free and G/N is torsion; cf. [7], Vol. II. We summarize these properties including parts of (2.2) into the

DEFINITION. A submodule N of the R-module G is a very nice submodule iff N is S-torsion-free with G/N a direct sum of S-bounded R-modules.

Let $\sigma = (s_n)$ and $\tau = (t_n)$ ($n \in \omega$) be two fixed null sequences of elements in S such that $s_0 = t_0 = 1$; and denote $\tau\sigma^{-1} = (t_n s_n^{-1})$. Then σ is monotone if $s_{n+1}R \subsetneqq s_n R$ for all $n \in \omega$.

Our main result will be the following

(2.4) THEOREM. *Let $T = \bigoplus_{n\in\omega} A/t_n A$. Then for each S-pure S-dense submodule N of $\hat{A}$ there exists an R-module $G(N) = G(\sigma, \tau, N) \subset \hat{T}$ such that*

(1) *$N \subset G(N)$ and $G(N)/N = \bigoplus_{m\in\omega} N/(s_{m+1}s_m^{-1}t_m N)$,*

(2) *$T \subseteq G(N)$ and $G(N)/T \cong S^{-1}N$,*

(3) *G is a functor and $f: N \to N'$ induces $G(f): G(N) \to G(N')$ fixing T,*

(4) $\operatorname{Hom}(G(N), G(N')) \cong \operatorname{Hom}(N, N') \oplus \operatorname{Hom}(G(N), T)$, *that is, if $\phi: G(N) \to G(N')$ then there is a unique $f: N \to N'$ with $\phi - G(f): G(N) \to T$.*

Using Corner's results [3] we get immediately

(2.5) COROLLARY. *Let A be a countable S-reduced and S-torsion-free R-algebra. Then we can find S-reduced mixed R-modules G_i ($i < 2^{\aleph_0}$) with the same* *torsion submodule T and S-divisible quotient G_i/T. The modules G_i contain very nice submodules N_i and we have* $\mathrm{Hom}(G_i, G_j) = A\delta_{ij} \oplus \mathrm{Hom}(G_i, T)$ *for all* $i, j < 2^{\aleph_0}$.

In section 4, which is devoted to mixed abelian groups, we will use the freedom of the choice of σ, τ to prescribe height sequences of the very nice subgroups N_i of G_i.

§3. Proof of the theorem

(3.1) *Construction of the R-modules* $G = G(\sigma, \tau, N)$ *and* $G' = G'(\sigma, \tau, N')$

Let $\sigma = (s_n)$ $(n \in \omega)$, $\tau = (t_n)$ $(n \in \omega)$ be the null sequences given by the theorem. Then we consider the R-module $T = \bigoplus_{n\in\omega} A/t_nA$. We will choose the module G to be a S-pure submodule of the S-adic completion $\hat{T}$ of T. The completion can be constructed in the cartesian product; so we have

(i) $$T = \bigoplus_{n\in\omega} A/t_nA \subset \hat{T} \subset \prod_{n\in\omega} (A/t_nA).$$

Let $\pi_n : \hat{A} \to \hat{A}/t_n\hat{A} = A/t_nA$ be the natural map. Consider the map $b = b^0 : \hat{A} \to \hat{T}$ with $b = \sum_{n=0}^{\infty} \pi_n$. In order to construct a S-pure submodule of $\hat{T}$ let $b^m : \hat{A} \to \hat{T}$ be the "tail"

$$b^m = \sum_{n=m}^{\infty} s_n s_m^{-1} \pi_n.$$

We derive the following equations:

(ii) $$\pi_n = b^n - s_n^{-1} s_{n+1} b^{n+1} \quad \text{and} \quad s_n b^n x \equiv bx \bmod T \qquad \text{for all } x \in \hat{A}.$$

Since $N \subseteq \hat{A}$, the expression $b^m N = b^m(N)$ has its natural meaning in T.

Let $G = G(N) = G(\sigma, \tau, N)$ be the R-module generated by all sets $b^m N$ $(m \in \omega)$. Then $G(\sigma, \tau, N) = \bigcup_{m\in\omega}(T \oplus b^m N)$. From (ii) we derive

(iii) $$T \subseteq G$$

and for each $g \in G$

(iv) there exist $k = k(g) \in \omega$, $u = u(g) \in N$ with $g \equiv b^k u \bmod T$.

First we establish condition (2) in the theorem, which is

(3.2) $G/T \cong S^{-1}N$, *and in particular T is the S-torsion submodule of G.*

Since $T \subseteq G$ by (3.1)(iii), it suffices to find an isomorphism $\phi : G/T \to S^{-1}N$. For any $g \in G$ choose $g \equiv b^k u \bmod T$ from (3.1)(iv), and let $(g + T)\phi = s_k^{-1} u$, which is a generic element of $S^{-1}N$. We want to show that ϕ is a well-defined map. If also $g+T=b^{k'}u'+T=g'+T$, then $b^k u - b^{k'}u' \in T$ and therefore we can find $i_0 \in \omega$ such that

$$e_i s_i (s_k^{-1} u - s_{k'}^{-1} u') = 0 \qquad \text{for all } i \geqq i_0.$$

This is equivalent to, say, that $s_i(s_k^{-1}u - s_{k'}^{-1}u') \in t_i \hat{A}$ and also $s_k^{-1}u - s_{k'}^{-1}u' \in t_i s_i^{-1}\hat{A}$ for all $i \geqq i_0$. Since $\hat{A}$ is S-reduced, we derive $s_k^{-1}u = s_{k'}^{-1}u'$ and therefore

$$(g+T)\phi = s_k^{-1}u = s_{k'}^{-1}u' = (g'+T)\phi,$$

i.e. ϕ is well-defined. Now it is easy to see that ϕ is an isomorphism.

Next we will consider homomorphisms from $G = G(\sigma,\tau,N)$ to $G^* = G(\sigma,\tau,N^*)$ where N and N^* are S-pure submodules of $\hat{A}$.

(3.3) *Any homomorphism $\phi : G \to G^*$ induces a homomorphism $\phi^* : N \to N^*$ such that*

$$(bu)\phi \equiv b(u\phi^*) \bmod T \qquad \textit{for all } u \in N$$

PROOF. From (3.1)(iv) we have

$$\text{(i)} \qquad (bu)\phi \equiv b^k u^* \bmod T \qquad \text{for some } k \in \omega, \quad u^* \in N.$$

If j is a fixed ordinal $< \omega$, we want to show that the j-th coordinate $(bu)\phi \restriction e_j$ of $(bu)\phi$ is contained in $e_j s_j \hat{A}$, i.e.

$$\text{(ii)} \qquad (bu)\phi \restriction e_j \in e_j s_j \hat{A}.$$

We break $bu = p_1 + p_2$ into two parts $p_1 = \sum_{i=0}^{j-1} e_i s_i u$ and $p_2 = \sum_{i=j}^{\infty} e_i s_i u$ and it is enough to show that $p_i\phi \restriction e_j \in e_j s_j A$. We consider first p_2 and observe from the definition of σ, τ that $s_i R \subseteq s_j R$ for all $i \geqq j$. Since $s_i \in R$, we have

$$p_2\phi \restriction e_j = \left(\sum_{i=j}^{\infty} e_i s_i u\right)\phi \restriction e_j = \sum_{i=j}^{\infty} (e_i s_i u)\phi \restriction e_j = \sum_{i=j}^{\infty} (e_i s_j^{-1} s_i u)\phi s_j \restriction e_j \in e_j s_j \hat{A}.$$

Next we determine $p_1\phi \restriction e_j$ and consider the contribution $d_i = (e_i s_i u)\phi \restriction e_j$ of the i-th summand $(i<j)$ to the j-th coordinate: Since $(t_i s_i^{-1})e_i s_i = t_i e_i = 0$ by definition of e_i, we also have $(t_i s_i^{-1})d_i = 0$. From $\mathrm{Ann}(e_j) = t_j \hat{A}$ we conclude $d_i \in e_j t_j s_i t_i^{-1} \hat{A}$. Since $\tau\sigma^{-1}$ is a descending null sequence as well, we have $t_j s_j^{-1} \in t_i s_i^{-1} R$ which implies $t_j s_i t_i^{-1} \in s_j R$. Therefore

$$d_i \in e_j t_j s_i t_i^{-1} \hat{A} \subseteq e_j s_j \hat{A} \quad \text{and} \quad p_1\phi \restriction e_j = \sum_{i=1}^{j-1} d_i \in e_j s_j \hat{A}$$

and (ii) follows.

Next we want to show that

(iii) there exists an element $u\phi^* \in N^*$ such that $(bu)\phi \equiv b(u\phi^*) \bmod T$.

By the definition of b, (i) and (ii) we find $j_0 \in \omega$ such that

$$e_j s_j s_k^{-1} u^* \in e_j s_j A \qquad \text{for all } j \geqq j_0,$$

and equivalently

$$e_j s_j s_k^{-1} u^* = e_j s_j y_j \qquad \text{for some } y_j \in \hat{A} \text{ and all } j \geqq j_0.$$

Since $\operatorname{Ann}(e_j) = t_j A$, the last equation is equivalent to, say, that

$$s_j(s_k^{-1}u^* - y_j) \in t_j A$$

which implies $u^* - s_k y_j \in s_k t_j s_j^{-1} A$. The algebra A is S-reduced and therefore in the limit $(j \to \infty)$ also $s_k y_j \to u^*$. Hence y_j $(j \in \omega)$ is a Cauchy sequence converging to some $u\phi^* \in \hat{A}$ and $u^* = s_k(u\phi^*) \in N^*$. Since N^* is an S-pure and S-torsion-free submodule of $\hat{A}$, we derive $u\phi^* \in N^*$ and (iii) follows from (i) since

$$(bu)\phi \equiv b^k u^* \equiv b^k s_k(u\phi^*) \equiv b(u\phi^*) \bmod T.$$

It is now easy to check that $\phi^*: N \to N^*$ is a well-defined homomorphism. □

The observation (3.3) has two trivial consequences, which are

(3.4) (a) $\operatorname{End} G = \operatorname{End}(N) \oplus \operatorname{Hom}(G, T)$,

(b) $\operatorname{Hom}(G, G^*) = \operatorname{Hom}(G, T)$.

Proof. Let $\phi \in \operatorname{Hom}(G, G^*)$ and $\phi^*: N \to N^*$ be as in (3.3). If $\operatorname{Hom}(N, N') = 0$, ϕ^* must vanish. From (3.3) we derive $(bN)\phi \subseteq T$ and since $T = tG'$ by (3.2), also $G\phi \subseteq T$ and (b) is shown.

Now consider the case $N = N^*$. The element $\phi^* \in \operatorname{End}(N)$ induces in a natural way an endomorphism of G which we denote ϕ^* again: $(b^m x)\phi^* = b^m(x\phi^*)$. If $g \in G$, let $g \equiv b^k u \bmod T$ as in (3.1)(iv). Since $T = tG$ is fully invariant, we obtain

$$(s_k g)(\phi - \phi^*) \equiv (s_k b^k u)\phi - s_k g\phi^* \equiv bu\phi - bu\phi^* = bu(\phi - \phi^*) \bmod T$$

from (3.1)(ii). However, from (3.3) we derive

$$(bu)\phi \equiv b(u\phi^*) \bmod T$$

and therefore $(s_k g)(\phi - \phi^*) \in T = tG$ and also $g(\phi - \phi^*) \in T$. We conclude $G(\phi - \phi^*) \subseteq T$ and (a) follows immediately.

Next we want to establish condition (1) of our theorem. Obviously $N \cong bN \subseteq G$ and therefore it is enough to prove

$$G/bN = \bigoplus_{m\in\omega} N/(s_{m+1}s_m^{-1}t_m)N. \tag{3.5}$$

Proof. Let $D_0 = 0$ and $D_m = \langle b^n N : n \leqq m\rangle / bN$ $(m \in \omega)$ which constitutes an ascending sequence of submodules of G/bN with union $D_\omega = \bigcup_{m\in\omega} D_m = G/bN$. It is easy to see that

$$(b^{m+1}N + bN)/bN \cong N/(s_{m+1}s_m^{-1}t_m)N$$

by definition of b, b^m; and obviously

$$D_{m+1} = D_m + (b^{m+1}N + bN)/bN. \tag{i}$$

We will show that the last sum is direct; then (3.5) will follow at once. Therefore, consider any $y + bN \in D_m \cap (b^{m+1}N + bN)/bN$. We may choose elements $u, u_i \in N$ $(i \leqq m)$ such that

$$y \equiv -b^{m+1}u \equiv \sum_{i=1}^{m} b^i u_i \bmod bN \tag{ii}$$

and also

$$b^{m+1}u + \sum_{i=0}^{m} b^i u_i = 0 \qquad (u, u_i \in N). \tag{iii}$$

Recall the definition of b^i and consider the m-th coordinate of the equation (iii). Since $b^{m+1}\restriction e_m = 0$ and $\mathrm{Ann}(e_m) = t_m A$, we have $\sum_{i=0}^{m} s_i^{-1}s_m u_i \in t_m\hat{A}$ and

$$\sum_{i=0}^{m} s_i^{-1}u_i \in t_m s_m^{-1}\hat{A} \qquad \text{where } t_m s_m^{-1} \in S. \tag{iv}$$

Now we evaluate (iii) at its $(m-1)$-th coordinate. Since $b^m \restriction e_{m-1} = b^{m+1}\restriction e_{m-1} = 0$ and $\mathrm{Ann}(e_{m-1}) = t_{m-1}\hat{A}$, we have

$$\sum_{i=0}^{m-1} s_i^{-1}s_{m-1}u_i \in t_{m-1}\hat{A} \quad \text{and} \quad \sum_{i=0}^{m-1} s_i^{-1}u_i \in t_{m-1}s_{m-1}^{-1}\hat{A}.$$

By hypothesis $\tau\sigma^{-1}$ is a monotone sequence, which implies $t_m s_m^{-1}\hat{A} \subseteq t_{m-1}s_{m-1}^{-1}\hat{A}$; subtracting the last formula from (iv) leads to

$$(s_m^{-1}u_m) \in t_{m-1}s_{m-1}^{-1}\hat{A} \qquad \text{with } t_{m-1}s_{m-1}^{-1} \in S.$$

Since N is a S-pure submodule of $\hat{A}$ containing u_m, we conclude $s_m^{-1}u_m \in N$. By induction on $i \leqq m$ we have $s_i^{-1} u_i \in N$ for $i \leqq m$ and therefore $u_i = s_i v_i$ for some $v_i \in N$ $(i \leqq m)$. Substitution of these elements into (iii) leads to

$$b^{m+1} u + \sum_{i=0}^{m} b^i s_i v_i = 0.$$

From (3.1)(ii) we see that this equation may be transformed into

$$\text{(v)} \qquad b^{m+1} u = bv + \sum_{i=0}^{m-1} e_i x_i \qquad \text{for some } v, x_i \in N.$$

Now we evaluate (v) at any coordinate $j \geqq m$. Therefore

$$s_j s_{m+1}^{-1} u \equiv s_j v \bmod t_j \hat{A}$$

and $u - s_{m+1} v \in t_j s_j^{-1} s_{m+1} \hat{A}$ $(j \geqq m)$, which implies in the limit $u = s_{m+1} v$. We transform (v) with (3.1)(ii) into

$$-\sum_{i=0}^{m} e_i s_i v = b^{m+1} u - bv = \sum_{i=0}^{m-1} e_i x_i .$$

We derive $e_m s_m v = 0$ evaluating the m-th coordinate of the last equation and equivalently $s_m v = t_m v^*$ for some $v^* \in A$. Therefore $u = s_{m+1} v = s_{m+1} s_m^{-1} t_m v^*$. However

$$(t_m s_m^{-1})(e_i s_i) = 0 \qquad \text{for all } i \leqq m,$$

which implies $b^{m+1} u = bv$ and $y \equiv -b^{m+1} u \equiv -bv \equiv 0 \bmod bN$ by (ii), i.e. (i) is a direct sum.

§4. Prescribing height sequences of mixed modules

Here we are interested in families of p-local countable mixed modules G with fixed torsion submodule T and fixed quotient G/T and a fixed algebra A such that $\operatorname{End} G = A \oplus \operatorname{Hom}(G, T)$. We derived the existence of $2^{\aleph_0}$ such mixed modules with pair-wise non-isomorphic very nice submodules; cf. (2.5). In (4.1) we will see that all height sequences of the different embeddings $N \subseteq G$ coincide.

Therefore, it is natural to ask whether such a family of $2^{\aleph_0}$ modules G with a *fixed* very nice submodule N may be found. This will be achieved by variation of the null sequence σ, which has the effect that the height sequences of the embedding $N \subseteq G$ no longer coincide.

We will first show how the height sequences depend on σ, τ. Recall a few elementary facts on height sequences of p-local abelian groups from [7], Vol. II,

p. 3 and [8], pp. 73,74. If g is an element of the abelian group G, then $h_p(g) = h(g)$ denotes its generalized height. Following Kaplansky, define the indicator or the height sequence of g to be $H(g) = (h(p^n g)\ (n \in \omega)$. Two height sequences of ordinals $\rho = (\rho_n)$, $\chi = (\chi_n)$ are equivalent iff there exist integers i, j such that $\rho_{i+n} = \chi_{j+n}$ for all $n \in \omega$. In general $\rho_{n+1} \geqq \rho_n + 1$ and ρ has a gap at n if this inequality is proper. The jump will be $\rho_{n+1} - \rho_n - 1$.

(4.1) LEMMA. *Let $R = Z$, $\sigma = (p^{s(n)})$, $\tau = (p^{t(n)})$ and $N = bN$ the nice subgroup of $G = G(\sigma, \tau, N)$ as in* (2.4). *Then the following holds:*

(a) *All height sequences of bN are equivalent.*

(b) *If $u \in N \backslash pN$, then $H(bu)$ has its only gaps at $n(k) = t(k) - s(k)$ $(k \in \omega)$ and the jump is $s(k+1) - s(k)$.*

REMARK. From (4.1) it is obvious that we can find groups with $2^{\aleph_0}$ inequivalent height sequences with respect to a fixed N.

PROOF. Since $s_k = p^{s(k)}$ and $t_k = p^{t(k)}$ we have from (3.1) that $T = \bigoplus_{n \in \omega} e_n A$ such that

$$\operatorname{Ann}(e_n) = p^{t(n)} A \quad \text{and} \quad b = \sum_{n \in \omega} e_n p^{s(n)} \in \bigoplus_{n \in \omega} e_n A.$$

If $u \in N \backslash pN$, $n \in \omega$ and k is minimal with $n < t(k) - s(k)$, then it is obvious that $h(bp^n u) = s(k) + n$. In the case $n = t(k) - s(k)$ we have

$$h(bp^{t(k)-s(k)} u) = s(k+1) - s(k) + t(k)$$

and similarly

$$h(bp^{t(k)-s(k)-1} u) = s(k).$$

Therefore (a), (b) follow at once.

(4.2) THEOREM. *Let $\sigma = (s_n)$, $\tau = (t_n)$, $\sigma' = (s'_n)$ be monotone null sequences in S such that $\tau\sigma^{-1}$ and $\tau\sigma'^{-1}$ are monotone null sequences and $\sigma\sigma'^{-1}$ contains a null sequence. Let N be a S-pure submodule of the completion $\hat{A}$ of a S-torsion-free and S-reduced R-algebra A such that* $\operatorname{End} N = A$. *Then the R-modules $G = G(\sigma, \tau, N)$ and $G' = G(\sigma', \tau, N)$ with torsion submodule T as in* (2.4) *satisfy the additional condition*

$$\operatorname{Hom}(G, G') = \operatorname{Hom}(G, T).$$

PROOF. Let $T = \bigoplus_{n \in \omega} e_n A$ and $b \in \hat{T}$ as in (3.1). Furthermore, let $b' = \sum e_n s'_n \in \hat{T}$ such that $N \cong b'N \subseteq G'$. If $v \in N$ and $\phi \in \operatorname{Hom}(G, G')$ we have

(i) $$\phi(bv) \equiv b'^k u \bmod T$$

from (3.1)(iv). As in (3.3)(ii) we conclude

(ii) $$\phi(bv)\restriction e_j \in e_j s_j \hat{A}.$$

By (i) and (ii) we find $j_0 \in \omega$ such that $e_j s'_j s'^{-1}_k u \in e_j s_j \hat{A}$ and equivalently

$$e_j(s'_j s'^{-1}_k u - s_j a_j) = 0 \qquad \text{for some } a_j \in \hat{A} \text{ and all } j \geqq j_0.$$

Therefore $u \equiv s_j s'^{-1}_j s'_k a_j \mod s'^{-1}_j s'_k t_j \hat{A}$ for all $j \geqq j_0$. Since $\sigma\sigma'^{-1} = (s_j s'^{-1}_j)$ by hypothesis contains a null sequence, we conclude $u = 0$. From (i) we derive $\phi \in \operatorname{Hom}(G, T)$.

(4.3) COROLLARY. *Let A, N and T be as in* (2.4). *Then we can find a family of* $2^{\aleph_0}$ *mixed modules $G_i = G_i(\sigma_i, \tau, N)$ with the following properties*:

(a) *$tG_i = T$, $G_i/T \cong S^{-1}N$, N is isomorphic to a very nice submodule $N_i \subseteq G_i$ and*

$$\operatorname{End} G_i = A \oplus \operatorname{Hom}(G_i, T) \qquad (i \in 2^{\aleph_0}).$$

(b) $\operatorname{Hom}(G_i, G_j) = \operatorname{Hom}(G_i, T)$ iff $i \neq j \in 2^{\aleph_0}$.

PROOF. The result will follow from (4.2) if we ensure that the sequences σ_i ($i \in 2^{\aleph_0}$) satisfy the required conditions. For this it is relevant that the sequences $\sigma_i\sigma_j^{-1}$ ($i \neq j$) contain a null sequence. If we consider the index set ω of these sequences, this will follow by our choice of sequences in ω:

Choose a sequence $\sigma = (s_n)$ such that $(t_n - s_n)$ is strictly increasing. Therefore we can find unbounded sequences (a_n) and (b_n) in ω such that the following holds:

$$t_n - s_n + a_n < t_{n+1} - s_{n+1} - b_{n+1},$$

$$s_n + b_n < s_{n+1} - a_{n+1},$$

and $(t_n - s_n - b_n)$ is strictly increasing.

Now let $\{T_i : i \in 2^{\aleph_0}\}$ be a family of almost disjoint subsets T_i of ω with $|T_i| = |\omega \setminus T_i| = \infty$. Then we decompose $T_i = T_i^0 \,\dot{\cup}\, T_i^1$ into infinite subsets and let $\sigma_i = (s_{in})$ be defined as

$$s_{in} = \begin{cases} s_n & \text{if } n \in \omega \setminus T_i, \\ s_n - a_n & \text{if } n \in T_i^0, \\ s_n + b_n & \text{if } n \in T_i^1. \end{cases}$$

The assumptions on s_n, t_n, a_n and b_n ensure that $(t_n - s_{in})$ is strictly increasing in n and $\sup_{n\in\omega}\{s_{in} - s_{jn}\} = \infty$.

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